Abstract Algebra II
Problem Set 2: Group Actions and Sylow

Exam style questions are marked with (*).
Questions marked Challenging need not be attempted.

Consider the following definitions of group action.

Definition. A *group action* of a group G on a set X is:

- a homomorphism $\alpha : G \to S_X$;
- a map $\alpha : G \times X \to X$ satisfying:
  - $\alpha(e, x) = x$ for all $x \in X$;
  - $\alpha\big(g, \alpha(h, x)\big) = \alpha(gh, x)$ for all $g, h \in G$ and $x \in X$.

Question 1. Prove that the two previous definitions of group actions are equivalent.

Question 2. For a group G and the set X of subgroups of G, consider the conjugation of elements of X by elements of G, i.e. $\alpha : G \times X \to X : (g, H) \mapsto H^g$.

- Prove that α is a group action.
- Compare the orbits of α with the conjugacy classes of the subgroups of G.
- Let $H \in X$ (i.e. $H < G$). The stabiliser $\mathrm{Stab}(H)$ of H under α is called the *normaliser of* H. State an equivalent condition on this stabiliser for H to be a normal subgroup of G.
- Prove that the normaliser of any $H \in X$ is normal in G.
- Find a bijection between $G/\operatorname{Stab}(H)$ and $\{H^g \mid g \in G\}$, the set of conjugate subgroups to H.

Question 3. When is a stabiliser of an element under a group action a *normal subgroup* of the group?

Question 4. Let G be a group acting on a set X, so we have a homomorphism $\varphi : G \to S_X$. Prove that this action is trivial ($gx = x$ for all $g \in G$) if and only if $\ker(\varphi) = G$.

Sylow's theorems

Question 5. Show that any group of the following order can not be simple:

- 686;
- 47;
- 36;
- 72.

Question 6.
1. Let G be a simple subgroup of S_6. Using the first isomorphism theorem, or otherwise, show that G must be a subgroup of A_6.
2. Using Part (1), or otherwise, prove that any group of order 120 must not be simple.

Challenging

Question 7. Let p be a prime and let $q \in \mathbb{N}$ be such that $\gcd(p, q) = 1$. Prove that p does not divide

$$\binom{p^n q}{p^n}$$

for any n.